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ORIGINAL PAPER

An efficient wavelet based spectral method to singular boundary value problems

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Abstract In this paper, we have established an efficient wavelet based approximation method to nonlinear singular boundary value problems. To the best of our knowledge, until now there is no rigorous shifted second kind Chebyshev wavelet (S2KCWM) solution has been addressed for the nonlinear differential equations in population biology. With the help of shifted second kind Chebyshev wavelets operational matrices, the nonlinear differential equations are converted into a system of algebraic equations. The convergence of the proposed method is established. The power of the manageable method is confirmed. Finally, we have given some numerical examples to demonstrate the validity and applicability of the proposed wavelet method.

Keywords Singular boundary value problems · Emden–Fowler equations · Chebyshev wavelets · Operational matrices

1 Introduction

In recent years, nonlinear singular boundary value problems (NSBVPs) arise in many branches of engineering and applied mathematics such as chemical reactions gas dynamics, electro-hydrodynamics, nuclear physics, atomic structures, atomic calculations, physiology and medical sciences. The numerical study of singular boundary value problems arising in various physical models has been done by many authors [1–11] and a variety of methods have been introduced to solve such singular boundary

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value problems [2,3,5,8–10]. Although, these numerical methods have many advantages, but a huge amount of computational work is needed.

Wavelet based spectral methods have been successfully introduced for solving non-linear type differential equations from the beginning of 1990s. In the last few decades the wavelet based approximation methods for such problems have attracted excellent attention and numerous papers about this topic have been published. Wavelets analysis possesses many useful properties, such as compact support, orthogonality, dyadic, orthonormality and multi-resolution analysis (MRA). An excellent discussion on wavelet transforms and the Fourier transforms presented by Gilbert Strang in the year 1993. In the numerical analysis, wavelet based methods and hybrid methods become important tools because of the properties of localization. In wavelet based techniques, there are two important ways of improving the approximation of the solutions: increasing the order of the wavelet family and the increasing the resolution level of the wavelet. There is a growing interest in using wavelets to study problems, of greater computational complexity. Wavelet methods have proved to be very effective and efficient tool for solving problems of mathematical calculus [10–29]. Among the wavelet transform families the Haar, Legendre wavelets and Chebyshev wavelets deserve much attention [25–29].

The basic idea of Chebyshev wavelet method (CWM) is to convert the differential equations to a system of algebraic equations by the operational matrices of integral or derivative. The main goal is to show how wavelets and multi-resolution analysis (MRA) can be applied for improving the method in terms of easy implementability and achieving the rapidity of its convergence. Wavelets, as very well-localized functions, are considerably useful for solving differential equations and provide accurate solutions. Also, the wavelet technique allows the creation of very fast algorithms when compared with the algorithms ordinarily used [25–28]. Recently, Hariharan and Kannan [29] reviewed the wavelet transforms methods for solving a few reaction-diffusion equations arising in science and engineering. In this paper, the shifted second kind Chebyshev wavelet method (S2KCWM) is applied to singular boundary value problems arising in biology. The method consists of reducing the differential equations to a set of algebraic equations by first expanding the candidate function as Chebyshev wavelets with unknown coefficients [12–25].
Consider the equation

$$\frac{d^2y}{dx^2} + \frac{\alpha}{x}\frac{dy}{dx} = f(x, y). \quad 0 \le x \le 1, \quad \alpha \ge 1 \tag{1}$$

with the boundary conditions

$$y'(0) = 0, \quad ay(1) + by'(1) = c$$

where a, b and c are real constants.

This paper is organized as follows. In Sect. 2, some properties of shifted second kind Chebyshev wavelets are presented. Some numerical examples are given in Sect. 3. Concluding remarks are given in Sect. 4.

2 Some properties of second kind Chebyshev polynomials and their shifted forms [25]

It is well known that the second kind Chebyshev polynomials are defined on $[-1, 1]$ by

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \quad x = \cos\theta. \tag{2}$$

These polynomials are orthogonal on $[-1, 1]$

$$\int_{-1}^{1} \sqrt{1-x^2} U_m(x) U_n(x) dx = \begin{cases} 0, & m \neq n \\ \frac{\pi}{2} & m = n \end{cases} \tag{3}$$

The following properties of second kind Chebyshev polynomials are of fundamental importance in the sequel. They are eigen functions of the following singular Sturm–Liouville equation.

$$(1-x^2) D^2 \varphi_k(x) - 3x D\varphi_k(x) + k(k+2)\varphi_k(x) = 0, \tag{4}$$

where $D \equiv \frac{d}{dx}$ and may be generated by using the recurrence relation

$$U_{k+1}(x) = 2x U_k(x) - U_{k-1}(x), \quad k = 1, 2, 3, \ldots \tag{5}$$

Starting from $\mathrm{U}_0(\mathrm{x}) = 1$ and $\mathrm{U}_1(\mathrm{x}) = 2\mathrm{x}$, or from Rodrigues formula

$$U_n(x) = \frac{(-2)^n\,(n+1)!}{(2n+1)!\sqrt{(1-x^2)}} D^n \left[(1-x^2)^{n+\frac{1}{2}} \right] \tag{6}$$

Theorem 2.1 *The first derivative of second kind Chebyshev polynomials is of the form*

$$DU_n(x) = 2 \sum_{\substack{k=0 \\ (k+n) odd}}^{n-1} (k+1) U_k(x). \tag{7}$$

Definition [25]: The shifted second kind Chebyshev polynomials are defined on [0,1] by $U_n^*(x) = U_n(2x-1)$. All results of second kind Chebyshev polynomials can be easily transformed to give the corresponding results for their shifted forms. The orthogonally relation with respect to the weight function $\sqrt{x-x^2}$ is given by

$$\int_{0}^{1} \sqrt{x-x^2} U_n^*(x) U_m^*(x) dx = \begin{cases} 0, & m \neq n \\ \frac{\pi}{8}, & m = n. \end{cases} \tag{8}$$

Corollary [30]: *The first derivative of the shifted second kind Chebyshev polynomial is given by*

$$DU_n^*(x) = 4 \sum_{\substack{k=0 \\ (k+n) odd}} (k+1)U_k^*(x) \tag{9}$$

2.1 Shifted Second kind Chebyshev operational matrix (S2KCOM) of derivatives [25]

Second kind Chebyshev wavelets are denoted by $\psi_{n,m}(t) = \psi(k, n, m, t)$, where k, n are positive integers and m is the order of second kind Chebyshev polynomials.

Here t is the normalized time. They are defined on the interval [0, 1] by

$$\psi_{n,m}(t) = \begin{cases} \frac{2^{\frac{k+3}{2}}}{\sqrt{\pi}} U_m^*(2^k t - n), & t \in \left[\frac{n}{2^k}, \frac{n+1}{2^k}\right], \\ 0 & otherwise \end{cases} \tag{10}$$

$m = 0, 1, \ldots M$, $n = 0, 1, \ldots 2^k - 1$. A function f (t) defined over [0, 1] may be expanded in terms second kind Chebyshev wavelets as

$$f(t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{nm} \psi_{nm}(t). \tag{11}$$

where

$$c_{nm} = (f(t), \psi_{nm}(t))_w = \int_0^1 \sqrt{t - t^2} f(t) \psi_{nm}(t) dt. \tag{12}$$

If the infinite series is truncated, then it can be written as

$$f(t) \simeq \sum_{n=0}^{2^k-1} \sum_{m=0}^{M} c_{nm} \psi_{nm}(t). = C^T \psi(t) \tag{13}$$

where C and ψ (t) are $2^k(M+1) \times 1$ defined by

$$\left.\begin{array}{l} C \quad\; = \left[c_{0,0}, c_{0,1}, \ldots c_{0,M}, \ldots, c_{2^k-1,M}, \ldots c_{2^k-1,1}, \ldots, c_{2^k-1,M}\right]^T \\ \psi(t) = [\psi_{0,0}, \psi_{0,1}, \ldots, \psi_{0,M}, \ldots \psi_{2^k-1,M}, \ldots, \psi_{2^k-1,1}, \ldots, \psi_{2^k-1,M}]^T \end{array}\right\} \tag{14}$$

A shifted second kind Chebyshev wavelets operational matrix of the first derivative is stated and proved in the following theorem.

Theorem 2.2 *Let* Ψ(t) *be the second kind Chebyshev wavelets vector defined in Eq. (10). Then the first derivative of the vector* Ψ(t) *can be expressed as*

$$\frac{d\psi(t)}{dt} = D\,\psi(t) \tag{15}$$

where D *is* 2^k(M+1) *square matrix of derivatives and is defined by*

$$D = \begin{bmatrix} F & O & . & . & . & O \\ O & F & . & . & . & O \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ O & O & . & . & . & F \end{bmatrix}$$

in which F *is an* (M+1) *square matrix and its* (r,s)th *element is defined by*

$$F_{r,s} = \begin{cases} 2^{k+2}s & r \geq 2, \quad r > s \quad and \quad (r+s) \quad odd. \\ 0, & otherwise \end{cases} \tag{16}$$

Corollary [25]: *The operational matrix for the* nth *derivative can be obtained from*

$$\frac{d^n \psi(t)}{dt^n} = D^n \psi(t), \quad n = 1, 2, \ldots where\ D^n\ is\ the\ nth\ power\ of\ D. \tag{17}$$

2.2 Linear second-order two-point boundary value problems [25]

Consider the linear second-order differential equation

$$y''(x) + g_1(x)y'(x) + g_2(x)y(x) = G(x), \quad x \in [0, 1], \tag{18}$$

Subject to the initial conditions

$$y(0) = \alpha, \quad y'(0) = \beta \tag{19}$$

(or) the boundary conditions

$$y(0) = \alpha, \quad y(1) = \beta \tag{20}$$

or the most general mixed boundary conditions

$$\alpha_1 y(0) + \alpha_2 y'(0) = \alpha, \quad b_1 y(1) + b_2 y'(1) = \beta. \tag{21}$$

If we approximate the functions y(x), g_1(x), g_2(x) and G(x) in terms of the second kind Chebyshev wavelet basis, one can write

$$y(x) \approx \sum_{n=0}^{2^k-1} \sum_{m=0}^{M} c_{nm}\psi_{nm}(x) = C^T \psi(x). \quad g_1(x) \approx \sum_{n=0}^{2^k-1} \sum_{m=0}^{M} g_{nm}\psi_{nm}(x) = G_1^T \psi(x) \tag{22}$$

$$g_2(x) \approx \sum_{n=0}^{2^k-1} \sum_{m=0}^{M} g_{nm}\psi_{nm}(x) = G_2^T \psi(x) \quad g(x) = \sum_{n=0}^{2^k-1} \sum_{m=0}^{M} g_{nm}\psi_{nm}(x) = G^T \psi(x) \tag{23}$$

Then

$$y'(x) \approx C^T D\psi(x), \quad y''(x) = C^T D^2\psi(x) \tag{24}$$

Now substitution of relations Eqs. (22), (23) and (24) into Eq. (18), enable us to define the residual, R(x), of this equation as

$$R(x) = C^T D^2\psi(x) + G_1^T\psi(x)(\psi(x))^T D^T C + G_2^T\psi(x)(\psi(x)^T C - G^T\psi(x). \tag{25}$$

and application of the tau method, yields the following $(2^k(M+1)-2)$ linear equations in the unknown expansion coefficients, c_{nm}, namely

$$\int_0^1 \sqrt{x-x^2}\psi_j(x)R(x)dx = 0, \quad j = 1, 2, \ldots 2^k(M+1) - 2. \tag{26}$$

Moreover, the initial conditions Eq. (19), the boundary conditions Eq. (20), and the mixed boundary conditions Eq. (21) lead respectively, to the following equations

$$\begin{array}{ll} C^T\psi(0) = \alpha, & \psi(0) = \beta, \\ C^T\psi(0) = \alpha & C^T\psi(1) = \beta \end{array} \tag{27}$$

and

$$a_1 C^T\psi(0) + a_2 D\psi(0) = \alpha, \quad b_1 C^T\psi(1) + b_2 C^T D\psi(1) = \beta \tag{28}$$

Thus Eq. (25) with the two equations of Eqs. (27) or (28) generate $2^k(M+1)$ a set of linear equations which can be solved for the unknown components of the vector C, and hence an approximate spectral wavelets solution to y(x) can be obtained.

2.3 Solving nonlinear second-order two-point boundary value problems by the S2KCWM [25]

Consider the nonlinear differential equation

$$y''(x) = F(x, g(x), y'(x)), \tag{29}$$

Subject to the initial conditions

$$y(0) = \alpha, \quad y'(0) = \beta \tag{30}$$

or the boundary conditions

$$y(0) = \alpha, \quad y(1) = \beta \tag{31}$$

or the most general mixed boundary conditions

$$\alpha_1 y(0) + \alpha_2 y'(0) = \alpha, \quad b_1 y(1) + b_2 y'(1) = \beta. \tag{32}$$

Using the aforesaid scheme, one can obtain

$$C^T D^2\psi(x) = F(x, g(x), C^T D\psi(x)) \tag{33}$$

To find an approximate solution to y(x), we compute Eq. (33) at the first $2^k(M+1)-2$ roots of $U^*_{2^k(M+1)}(x)$.

Eq. (33) with the two Eqs. (32) or (31) or (33) generate $2^k(M+1)$ non linear equations in the expansion coefficients C_{nm} which can be solved with the aid of Newton's iterative method.

2.4 Convergence theorem [25]

A function defined over [0, 1) may be expressed as

$$f(t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{nm}\psi_{nm}(t)$$

where

$$c_{nm} = < f(t), \psi_{nm}(t) >,$$

in which $< .,. >$ represents the inner product in $L^2[0,1]$.

Proof (See Ref. [12]). □

3 Numerical examples

Example 3.1 Now we consider the nonlinear singular boundary value problem which arises in the study of distribution of heat sources in the human head [2]. This is also known as Emden–Fowler equation of the second kind.

$$\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} = -e^{-y} \tag{34}$$

With the boundary conditions

$$y'(0) = 0, \quad y(1) + y'(1) = 0 \tag{35}$$

We solve Eq. (34) using the second kind shifting Chebyshev wavelets (S2KCW) algorithm described in Sects. 2.1–2.3 for the case corresponds to $M = 2$ and $k = 0$ to obtain the approximate solutions of $y(x)$.

First if we make the use of Eqs. (13) and (14) then the two operational matrices D and D^2 are given by

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 0 & 8 & 0 \end{pmatrix} \text{ and } D^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 32 & 0 & 0 \end{pmatrix}$$

Then, the function $\psi(r)$ can be obtained by using the following relation.

$$\psi(r) = \sqrt{\frac{2}{\pi}} \begin{pmatrix} 2 \\ 8r-4 \\ 32r^2-32r+6 \end{pmatrix} \tag{36}$$

If we set

$$C = \left(c_{0,0}, c_{0,1}, c_{0,2}\right)^T = \sqrt{\frac{\pi}{2}}\,(c_0, c_1, c_2)^T \tag{37}$$

then the shifted second kind Chebyshev wavelet scheme is given by

$$C^T D^2\psi(x) + \frac{2}{x}C^T D\psi(x) + \exp(-C^T\psi(x)) = 0 \tag{38}$$

If we select a root $r = \frac{2-\sqrt{2}}{4}$, then we get the following equation

$$16c_1 - 35.8752c_2 + 0.14645\left(e^{-(2c_0-2.828c_1+2c_2)}\right) = 0 \tag{39}$$

Using the boundary conditions, we gain

$$\begin{aligned} 8c_1 - 32c_2 &= 0 \\ 2c_0 + 12c_1 + 38c_2 &= 0 \end{aligned}$$

On solving these equations, we get

$$\begin{aligned} c_0 &= 0.1578 \\ c_1 &= -0.0147 \\ c_2 &= -0.0037 \end{aligned}$$

Now

$$y(x) = -0.1184x^2 + 0.3522 \tag{40}$$

Example 3.2 We consider the above nonlinear singular boundary value problem [2] with different boundary conditions (Fig. 1; Table 1)

$$\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} = -e^{-y} \tag{41}$$

With the boundary conditions

$$y'(0) = 0, \quad 2y(1) + y'(1) = 0 \tag{42}$$

then the SK2CW scheme is given by

$$C^T D^2\psi(x) + \frac{2}{x}C^T D\psi(x) + \exp(-C^T\psi(x)) = 0 \tag{43}$$

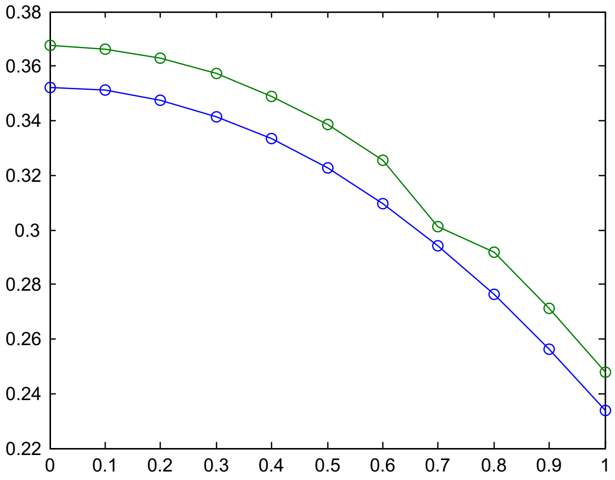

Fig. 1 Comparison of present method and method in [2]

Table 1 Approximate solutions for Example 3.1

x	Present method ($k = 0$)	Present method ($k = 1$)	Method in [2]
0.0000	0.3522	0.3672	0.3675
0.1000	0.3510	0.3663	0.3664
0.2000	0.3475	0.3622	0.3629
0.3000	0.3415	0.3553	0.3571
0.4000	0.3333	0.3478	0.3489
0.5000	0.3226	0.3384	0.3384
0.6000	0.3096	0.3253	0.3254
0.7000	0.2942	0.3013	0.3010
0.8000	0.2764	0.2922	0.2920
0.9000	0.2562	0.2712	0.2713
1.0000	0.2238	0.2479	0.2479

If we select a root $r = \frac{2-\sqrt{2}}{4}$, then we get the following equation

$$16c_1 - 35.8752c_2 + 0.14645\left(e^{-(2c_0-2.828c_1+2c_2)}\right) = 0 \tag{44}$$

Using the boundary conditions, we gain

$$8c_1 - 32c_2 = 0$$
$$4c_0 + 16c_1 + 44c_2 = 0$$

On solving these equations, we get

$$c_0 = 0.1089$$
$$c_1 = -0.01613$$
$$c_2 = -0.004034$$

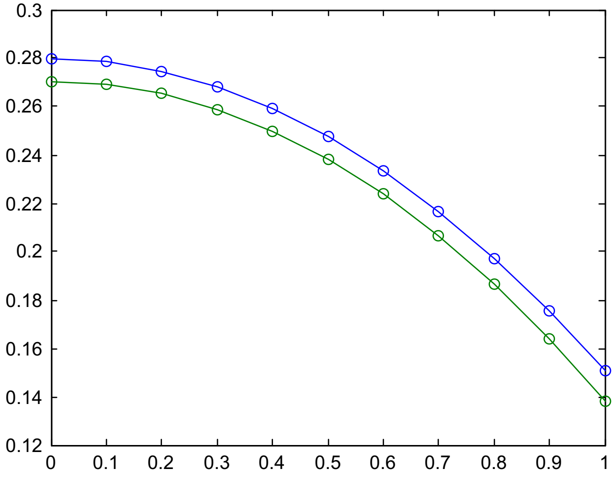

Fig. 2 Comparison of present method and method in [2]

Table 2 Approximate solutions for Example 3.2

x	Present method	Method in [2]
0.0000	0.2799	0.2704
0.1000	0.2786	0.2691
0.2000	0.2747	0.2653
0.3000	0.2683	0.2589
0.4000	0.2592	0.2499
0.5000	0.2476	0.2382
0.6000	0.2334	0.2239
0.7000	0.2166	0.2068
0.8000	0.1973	0.1868
0.9000	0.1753	0.1639
1.0000	0.1508	0.1379

Now

$$y(x) = -0.1291x^2 + 0.2799 \tag{45}$$

Example 3.3 Consider the another singular boundary value problem [4] (Fig. 2; Table 2)

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} = -e^y \tag{46}$$

With the boundary conditions

$$y'(0) = 0, \quad y(1) = 0 \tag{47}$$

The exact solution is

$$y(x) = 2\ln\left(\frac{c+1}{cx^2+1}\right),$$

where

$$c = 3 - 2\sqrt{2}$$

then the shifted second kind Chebyshev wavelet scheme is given by

$$C^T D^2\psi(x) + \frac{1}{x}C^T D\psi(x) + \exp(C^T\psi(x)) = 0 \tag{48}$$

If we select a root $r = \frac{2-\sqrt{2}}{4}$, then we get the following equation

$$8c_1 - 13.2512c_2 + 0.14645\left(e^{(2c_0-2.828c_1+2c_2)}\right) = 0 \tag{49}$$

Using the boundary conditions, we gain

$$8c_1 - 32c_2 = 0$$
$$2c_0 + 4c_1 + 6c_2 = 0$$

On solving these equations, we get

$$c_0 = 0.1214$$
$$c_1 = -0.0441$$
$$c_2 = -0.0110$$

Now

$$y(x) = -0.353x^2 + 0.353 \tag{50}$$

Example 3.4 We consider another singular boundary value problem which arises in the study of Steady- state oxygen diffusion in a spherical cell [5]

$$\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} = \frac{0.76129y}{y+0.03119} \tag{51}$$

With the boundary conditions

$$y'(0) = 0, \quad 5y(1) + y'(1) = 5 \tag{52}$$

then the shifted second kind Chebyshev wavelet scheme is given by

$$C^T D^2\psi(x) + \frac{2}{x}C^T D\psi(x) - \frac{0.76129C^T\psi(x)}{C^T\psi(x) + 0.03119} = 0 \tag{53}$$

If we select a root $r = \frac{2-\sqrt{2}}{4}$, then we get the following equation

$$\begin{aligned} &-0.223c_0 + 0.8143c_1 - 1.342c_2 - 45.248c_1^2 - 71.750c_2^2 + 32c_0c_1 \\ &\quad - 71.750c_0c_2 + 133.4553c_1c_2 = 0 \end{aligned} \tag{54}$$

Using the boundary conditions, we gain

$$\begin{aligned} &8c_1 - 32c_2 = 0 \\ &10c_0 + 28c_1 + 62c_2 = 0 \end{aligned}$$

On solving these equations, we gain

$$\begin{array}{ll} c_0 = 0.4335 & c_0 = 0.09076 \\ c_1 = 0.0153 & c_1 = 0.09408 \\ c_2 = 0.0038 & c_2 = 0.02352 \end{array}$$

Now we get two solutions respectively for the two set of values which satisfy boundary conditions (Fig. 3; Table 3)

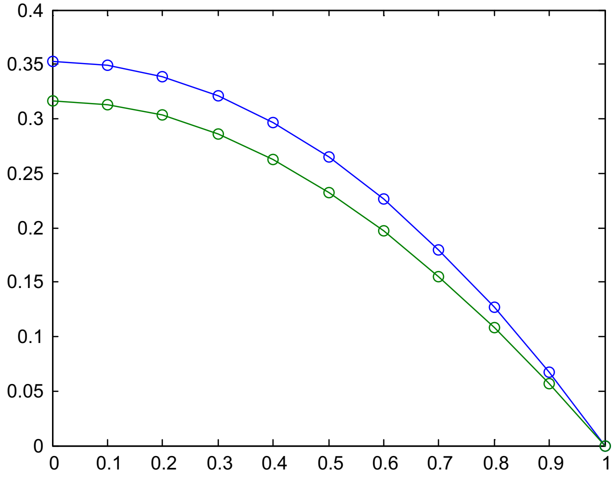

Fig. 3 Comparison of exact and present method solutions

Table 3 Numerical errors for Example 3.3

x	Exact	Present method	Absolute numerical error
0.0000	0.3167	0.3530	0.0363
0.1000	0.3133	0.3495	0.0362
0.2000	0.3031	0.3389	0.0358
0.3000	0.2861	0.3212	0.0351
0.4000	0.2626	0.2965	0.0339
0.5000	0.2327	0.2647	0.0320
0.6000	0.1969	0.2259	0.0290
0.7000	0.1553	0.1800	0.0247
0.8000	0.1083	0.1271	0.0188
0.9000	0.0564	0.0671	0.0107
1.0000	0.0000	0.0000	0.0000

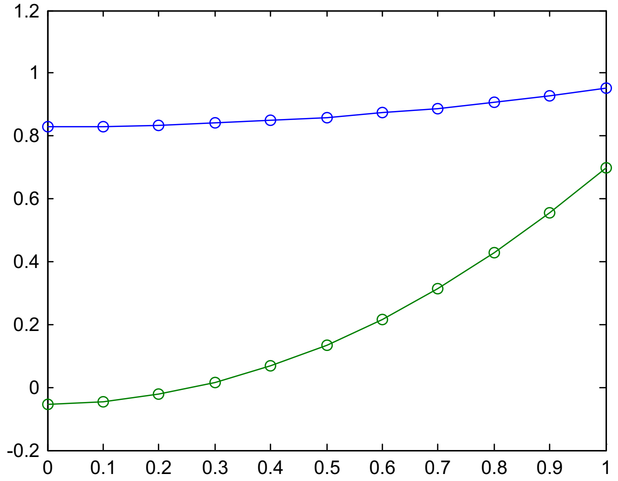

Fig. 4 Comparison of Eqs. (55) and (56)

$$y(x) = 0.1223x^2 + 0.8288 \tag{55}$$

$$y(x) = 0.75264x^2 - 0.05368 \tag{56}$$

From the figure we can conclude that Eq. (55) is the suitable solution for Example 3.4 (Fig. 4; Table 4).

Example 3.5 Consider the nonlinear singular boundary value problem describing the equilibrium of isothermal gas sphere (Fig. 5)

$$\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + y^5 = 0, \quad 0 < x < 1 \tag{57}$$

Table 4 Approximate solutions for Example 3.4

x	Present method ($k = 0$)	Present method ($k = 1$)	Method in [5]
0	0.8288	0.8284	0.8285
0.2	0.8337	0.8334	0.8334
0.4	0.8484	0.8480	0.8481
0.6	0.8728	0.8724	0.8725
0.8	0.9071	0.9067	0.9068
1.0	0.9511	0.9511	0.9511

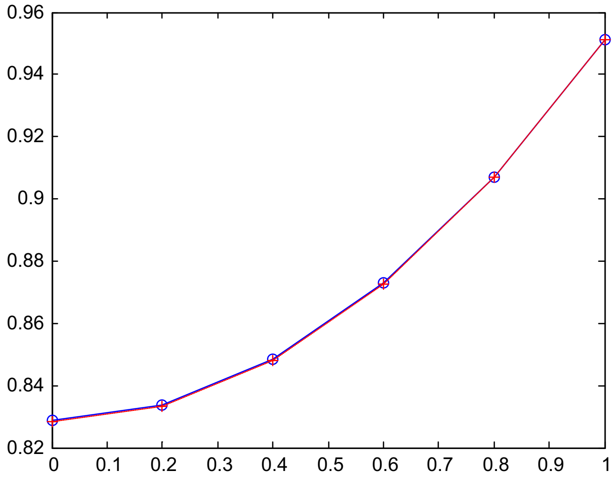

Fig. 5 Comparison of present method and method in [5]

With the boundary conditions

$$y'(0) = 0, \quad y(0) = 1 \tag{58}$$

The exact solution is

$$y(x) = \sqrt{\frac{3}{3 + x^2}}$$

This is also known as Emden–Fowler equation of the first kind.
then the shifted second kind Chebyshev wavelet scheme is given by

$$C^T D^2 \psi(x) + \frac{2}{x} C^T D\psi(x) + (C^T \psi(x))^5 = 0 \tag{59}$$

If we select a root $r = \frac{2-\sqrt{2}}{4}$, then we get the following equation

$$16c_1 - 35.8752c_2 + 0.14645(2c_0 - 2.828c_1 + 2c_2)^5 = 0 \tag{60}$$

Table 5 Numerical errors for Example 3.5

x	Exact	Present method	Absolute numerical error
0	1.0000	0.9998	0.0002
0.2	0.9934	0.9933	0.0001
0.4	0.9744	0.9736	0.0008
0.6	0.9449	0.9408	0.0041
0.8	0.9078	0.8949	0.0129
1.0	0.8660	0.8358	0.0302

Using the boundary conditions, we gain

$$8c_1 - 32c_2 = 0$$
$$2c_0 - 4c_1 + 6c_2 = 1$$

On solving these equations, we get

$$c_0 = 0.47442$$
$$c_1 = -0.02050$$
$$c_2 = -0.00512$$

Other four sets of solutions are complex numbers (Table 5).

Now

$$y(x) = 0.164x^2 - 0.99984 \tag{61}$$

Example 3.6 Consider the another nonlinear singular boundary value problem which arises in the radial stress on a rotationally symmetric shallow membrane cap [6] (Fig. 6)

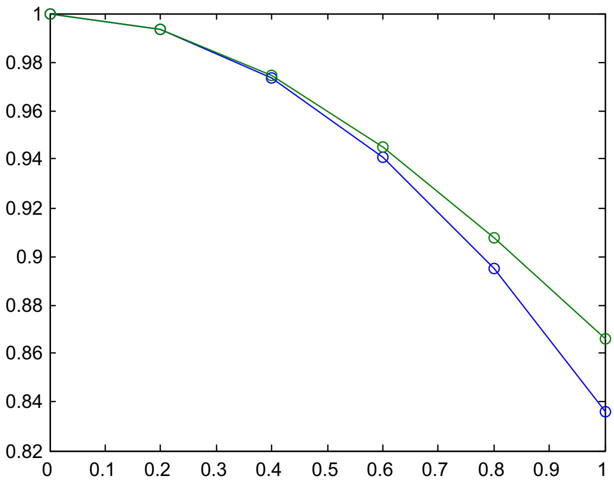

Fig. 6 Comparison of exact and present method solutions for Example 5

Table 6 Approximate solutions for Example 3.6

x	Present method	Method in [7]	Method in [6]
0.0	0.9546	0.9541	0.9522
0.1	0.9551	0.9546	0.9526
0.2	0.9564	0.9559	0.9541
0.3	0.9587	0.9582	0.9565
0.4	0.9619	0.9614	0.9599
0.5	0.9660	0.9655	0.9642
0.6	0.9710	0.9705	0.9695
0.7	0.9769	0.9765	0.9757
0.8	0.9837	0.9834	0.9829
0.9	0.9914	0.9912	0.9910
1.0	1.0000	1.0000	1.0000

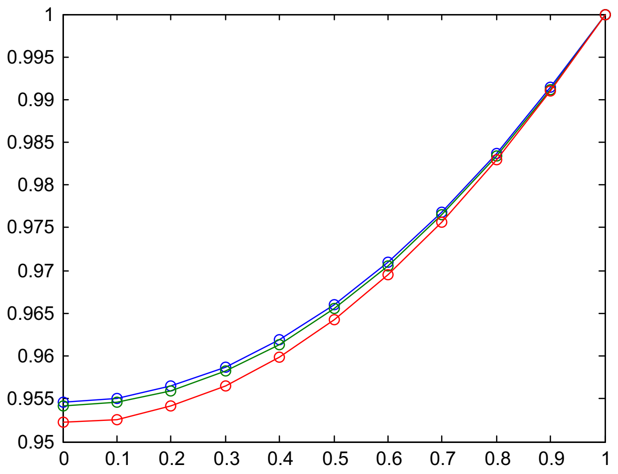

Fig. 7 Comparison of method in [7] and method in [6] and present method

$$\frac{d^2y}{dx^2} + \frac{3}{x}\frac{dy}{dx} = \frac{1}{2} - \frac{1}{8y^2(x)}, \quad 0 \le x \le 1 \tag{62}$$

With the boundary conditions

$$y'(0) = 0, \quad y(1) = 1 \tag{63}$$

then the shifted second kind Chebyshev wavelet scheme is given by

$$C^T D^2\psi(x) + \frac{3}{x}C^T D\psi(x) - \frac{1}{2} + \frac{1}{8(C^T\psi(x))^2} = 0 \tag{64}$$

If we select a root $r = \frac{2-\sqrt{2}}{4}$, then we get the following equations

$$\begin{aligned} &512c_2(2c_0 - 2.828c_1 + 2c_2)^2 + 163.8785(2c_0 - 2.828c_1 + 2c_2)^2(8c_1 - 22.624c_2) \\ &\quad - 4(2c_0 - 2.828c_1 + 2c_2)^2 = -1 \qquad (65) \end{aligned}$$

Using the boundary conditions, we gain

$$\begin{aligned} 8c_1 - 32c_2 &= 0 \\ 2c_0 + 4c_1 + 6c_2 &= 1 \end{aligned}$$

On solving these equations, we get

$$\begin{aligned} c_0 &= 0.484400 \\ c_1 &= 0.005672 \\ c_2 &= 0.001418 \end{aligned}$$

Other two sets of solutions are complex numbers (Table 6; Fig. 7).
Now

$$y(x) = 0.045376x^2 + 0.95462 \qquad (66)$$

4 Conclusion

In this paper, an efficient Chebyshev wavelet based approximation algorithm is presented for solving nonlinear singular boundary value problems. Numerical results show that the shifted second kind Chebyshev wavelet method (SK2CWM) can match the analytical solution very efficiently with quite a few calculations. Also the proposed method has a simple implementation process. Advantage of this algorithm is that high accurate approximate solutions are achieved using a few number of terms of the approximate expansion. The proposed scheme is the capability to overcome the difficulty arising in calculating the integral values while dealing with nonlinear problems. In addition the calculations involved in S2KCWM are simple, straight forward and low computation cost.

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References

1. S. Lin, Oxygen diffusion in s spherical cell with nonlinear oxygen uptake kinetics. J. Theor. Biol. **60**(2), 449–457 (1976)

2. R. Duggan, A. Goodman, Pointwise bounds for a nonlinear heat conduction model of the human head. Bull. Math. Biol. **48**(2), 229–236 (1986)
3. R.K. Pandey, Arvind K. Singh, On the convergence of finite difference method for a class of singular boundary value problems arising in physiology. J. Comput. Appl. Math. **166**, 553–564 (2004)
4. S. Khuri, A. Safy, A novel approach for the solution of a class of singular boundary values problems arising in physiology. Math. Comput. Model. **52**(3), 626–636 (2010)
5. A. Wazwaz, The variational iteration method for solving nonlinear singular boundary problems arising in various physical models. Commun. Nonlinear Sci. Numer. Simul. **16**(10), 3881–3886 (2011)
6. A. Ravikanth, K. Aruna, He's variational iteration method for treating nonlinear singular boundary value problems. Comput. Math. Appl. **60**(3), 821–829 (2010)
7. R. Singh, J. Kumar, An efficient numerical technique for the solution of nonlinear singular boundary value problems. Comput. Phys. Commun. **185**(4), 1282–1289 (2014)
8. H. caglar, N. caglar, M. Ozer, B-spline solution of non linear singular boundary value problems arising in physiology. Chaos Solitons Fractals **39**, 1232–1237 (2009)
9. R. Singh, J. Kumar, Solving a class of singular two-point boundary value problems using new modified decomposition method. ISRN Comput. Math. **2013**, 1–11 (2013)
10. Z. Cen, Numerical method for a class of singular non linear boundary value problems using green's functions. Int. J. Comput. Math. **24**(3–4), 29–310 (1988)
11. M. Inc, M. Engut, Y. Cherrualt, A different approach for solving singular two-point boundary value problems, kybernetes. Int. J. Syst. Cybern. **34**(7), 934–940 (2005)
12. H. Adibi, P. Assari, Chebyshev wavelet method for numerical solution of Fredholm Integral equations of the first kind. Math. Probl. Eng. **2010**, 138408 (2010)
13. Y. Li, Solving a nonlinear fractional differential equation using Chebyshev wavelets. Commun. Nonlinear Sci. Numer. Simul. **11**, 2284–2292 (2011)
14. S. Sohrabi, Comparison Chebyshev wavelet method with BPFs method for solving Abel's integral equation. Ain Shams Eng. J. **2**, 249–254 (2011)
15. J.C. Mason, C. David, *Handscomb, Chebyshev Polynomials* (Taylor and Francis, London, 2002)
16. M. Ghasemi, M. Avassoli Kajani, Numerical solution of time-varying delay systems by Chebyshev wavelets. Appl. Math. Model. **35**, 5235–5244 (2011)
17. W.M. Abd-Elhameed, E.H. Doha, Y.H. Youssri, New wavelets collocation method for solving second-order multipoint boundary value problems using Chebyshev polynomials of third and fourth kinds. Abstr. Appl. Anal. **2013**, 542839 (2013)
18. W.M. Abd-Elhameed, E.H. Doha, Y.H. Youssri, New spectral second kind Chebyshev wavelets algorithm for solving linear and nonlinear second-order differential equations involving singular and Bratu type equations. Abstr. Appl. Anal. **2013**, 715756 (2013)
19. SGh Hosseini, A new Operational Matrix of Derivative for Chebyshev wavelets and its applications in solving ordinary differential equations with non analytic solution. Appl. Math. Sci. **5**(51), 2537–2548 (2011)
20. A. Barzkar, M.K. Oshagh, P. Assari, M.A. Mehrpouya, Numerical solution of the nonlinear Fredholm integral equation and the Fredholm integro-differential equation of second kind using Chebyshev wavelets. World Appl. Sci. J. **18**(12), 1774–1782 (2012)
21. E. Babolian, F. Fattahzadeh, Numerical solution of differential equations by using Chebyshev wavelet operational matrix of integration. Appl. Math. Comput. **188**(1), 417–426 (2007)
22. L. Zhu, Q. Fan, Solving fractional nonlinear Fredholm integro-differential equations by the second kind Chebyshev wavelet. Commun. Nonlinear Sci. Numer. Simul. **17**(6), 2333–2341 (2012)
23. E.H. Doha, W.M. Abd-Elhameed, Y.H. Youssri, Second kind Chebyshev operational matrix algorithm for solving differential equations of Lane–Emden type. New Astron. **23–24**, 113–117 (2013)
24. E.H. Doha, W.M. Abd-Elhameed, M.A. Bassuony, New algorithms for solving high even-order differential equations using third and fourth Chebyshev–Galerkin methods. J. Comput. Phys. **236**, 563–579 (2013)
25. G. Hariharan, An efficient wavelet based approximation method to water quality assessment model in a uniform channel. Ains Shams Eng. J. **5**, 525–532 (2014)
26. G. Hariharan, K. Kannan, K.R. Sharma, Haar wavelet in estimating the depth profile of soil temperature. Appl. Math. Comput. **210**, 119–225 (2009)
27. G. Hariharan, K. Kannan, Haar wavelet method for solving Fisher's equation. Appl. Math. Comput. **211**, 284–292 (2009)

28. G. Hariharan, K. Kannan, A comparative study of a Haar wavelet method and a restrictive Taylor's series method for solving convection–diffusion equations. Int. J. Comput. Methods Eng. Sci. Mech. **11**(4), 173–184 (2010)
29. G. Hariharan, K. Kannan, Review of wavelet methods for the solution of reaction–diffusion problems in science and engineering. Appl. Math. Model. **38**(1), 799–813 (2014)